SENSOR FUSION

Conditional Gaussian Distribution (Lemma 7.1)

Sensor Fusion

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Conditional Gaussian Distribution

Lemma 7.1

If X and Y are two jointly distributed Gaussian stochastic variables according to

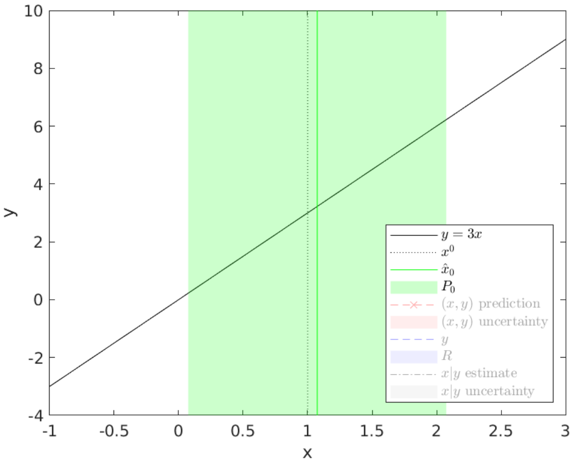
$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} P_{XX} & P_{XY} \\ P_{YX} & P_{YY} \end{pmatrix}\right),$$

then the conditional distribution of X, given the observed value of $Y = y$, is Gaussian distributed according to

$$(X|Y = y) \sim \mathcal{N}(\mu_X + P_{XY}P_{YY}^{-1}(y - \mu_Y), P_{XX} - P_{XY}P_{YY}^{-1}P_{YX}).$$

Illustrating Example

- Measurements: $y = 3x + e$ where $e \sim \mathcal{N}(0, R)$, $R = \frac{1}{2^2}$
- Actual x is $x^0 = 1$
- Prior: $x \sim \mathcal{N}(\hat{x}_0, P_0)$, $\hat{x}_0 = 1.1$, $P_0 = 1$



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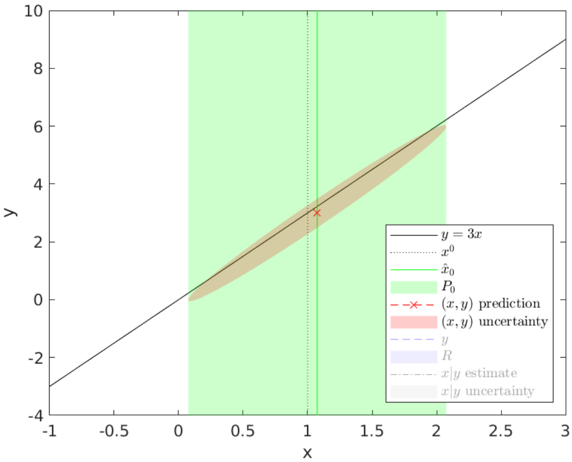
- Joint distribution

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \hat{x}_0 \\ 3\hat{x}_0 \end{pmatrix}, \begin{pmatrix} P_0 & 3P_0 \\ 3P_0 & 9P_0 + R \end{pmatrix}\right)$$



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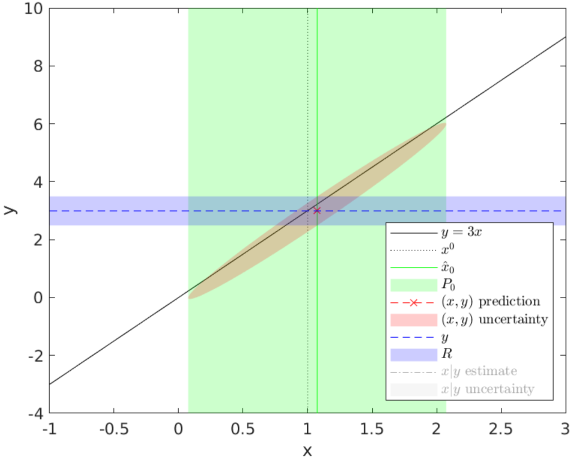
- Obtained measurement: $y = 2.99$



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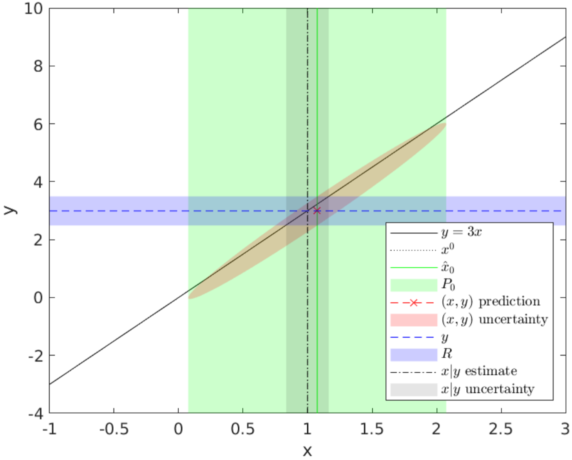
- Obtained measurement: $y = 2.99$
- Conditional distribution

$$(x|y) \sim \mathcal{N}(\hat{x}_0 + \frac{3P_0}{9P_0+R}(y - 3\hat{x}_0), \frac{R}{9P_0+R})$$



Proof of Lemma (1/4)

For simplicity, let $\tilde{x} = x - \mu_x$ and $\tilde{y} = y - \mu_y$. Now, compute the **conditional distribution**

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \mathcal{N}\left(\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}; \begin{pmatrix}x\\ y\end{pmatrix}, \begin{pmatrix}P_{XX} & P_{XY}\\ P_{YX} & P_{YY}\end{pmatrix}\right) \Big/ \mathcal{N}(y; \tilde{y}, P_{YY})$$

$$= \frac{\exp\left(\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}^T \begin{pmatrix}P_{XX} & P_{XY}\\ P_{YX} & P_{YY}\end{pmatrix}^{-1} \begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}\right)}{\sqrt{\det\left(2\pi \begin{pmatrix}P_{XX} & P_{XY}\\ P_{YX} & P_{YY}\end{pmatrix}\right)}} \Bigg/ \frac{\exp(\tilde{y}^T P_{YY}^{-1} \tilde{y})}{\sqrt{\det(2\pi P_{YY})}}$$

Proof of Lemma (2/4)

Block LDL decomposition of the joint covariance matrix yields

$$\begin{pmatrix} P_{XX} & P_{XY} \\ P_{YX} & P_{YY} \end{pmatrix} = \begin{pmatrix} I & P_{XY}P_{YY}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} P_{XX} - P_{XY}P_{YY}^{-1}P_{YX} & 0 \\ 0 & P_{YY} \end{pmatrix} \begin{pmatrix} I & P_{XY}P_{YY}^{-1} \\ 0 & I \end{pmatrix}^T.$$

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For notational convenience, let

$P = P_{XX} - P_{XY}P_{YY}^{-1}P_{YX}$

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$$\begin{pmatrix} P_{XX} & P_{XY} \\ P_{YX} & P_{YY} \end{pmatrix} = \begin{pmatrix} I & K \\ 0 & I \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P_{YY} \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}^T .$$

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For notational convenience, let
$P = P_{XX} - P_{XY} P_{YY}^{-1} P_{YX}$ and $K = P_{XY} P_{YY}^{-1}$.

Now the product rule can be used to compute the determinant for the denominator:

$$\det\left(\begin{pmatrix} P_{XX} & P_{XY} \\ P_{YX} & P_{YY} \end{pmatrix}\right) = \det\left(\begin{pmatrix} I & K \\ 0 & I \end{pmatrix}\right) \det\left(\begin{pmatrix} P & 0 \\ 0 & P_{YY} \end{pmatrix}\right) \det\left(\begin{pmatrix} I & K \\ 0 & I \end{pmatrix}^T\right)$$

$$= 1 \cdot \det(P) \det(P_{YY}) \cdot 1$$

Proof of Lemma (3/4)

Calculate the inverse of the joint covariance matrix,

$$\begin{pmatrix} P_{XX} & P_{XY} \\ P_{YX} & P_{YY} \end{pmatrix}^{-1} = \left(\begin{pmatrix} I & K \\ 0 & I \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P_{YY} \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}^T \right)^{-1}.$$

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The exponent cam now be expressed as

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}^T \begin{pmatrix} P_{XX} & P_{XY} \\ P_{YX} & P_{YY} \end{pmatrix}^{-1} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}^T \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix}^T \begin{pmatrix} P^{-1} & 0 \\ 0 & P_{YY}^{-1} \end{pmatrix} \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

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And define $\bar{x} = \tilde{x} - K\tilde{y}$ and substitute in.

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And define $\bar{x} = \tilde{x} - K\tilde{y}$ and substitute in.

Proof of Lemma (4/4)

Now the conditional distribution can be calculated

$$p_{X|Y}(x|y) = \frac{\exp\left(\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}^T \begin{pmatrix}P_{XX} & P_{XY}\\ P_{YX} & P_{YY}\end{pmatrix}^{-1} \begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}\right)}{\sqrt{\det\left(2\pi \begin{pmatrix}P_{XX} & P_{XY}\\ P_{YX} & P_{YY}\end{pmatrix}\right)}} \Bigg/ \frac{\exp(\tilde{y}^T P_{YY}^{-1} \tilde{y})}{\sqrt{\det(2\pi P_{YY})}}.$$

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$$p_{X|Y}(x|y) = \frac{\exp\left(\bar{x}^T P^{-1} \bar{x} + \tilde{y}^T P_{YY}^{-1} \tilde{y}\right)}{\sqrt{\det(2\pi P) \det(2\pi P_{YY})}} \Bigg/ \frac{\exp(\tilde{y}^T P_{YY}^{-1} \tilde{y})}{\sqrt{\det(2\pi P_{YY})}}.$$

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Now the conditional distribution can be calculated

$$p_{X|Y}(x|y) = \frac{\exp\left(\bar{x}^T P^{-1} \bar{x}\right)}{\sqrt{\det(2\pi P)}}.$$

Finally, expanding $\bar{x}$ yields,

$$\begin{aligned} \bar{x} &= \tilde{x} - K\tilde{y} = x - \mu_X - K(y - \mu_Y) = x - (\mu_X + K(y - \mu_Y)) \\ K &= P_{XY} P_{YY}^{-1} \\ P &= P_{XX} - P_{XY} P_{YY}^{-1} P_{YX} \end{aligned}$$

which shows that $p_{X|Y}(x|y)$ is a Gaussian distribution such that:

$$(X|Y = y) \sim \mathcal{N}(\mu_X + P_{XY} P_{YY}^{-1}(y - \mu_Y), P_{XX} - P_{XY} P_{YY}^{-1} P_{YX}).$$

Summary

Lemma 7.1

If X and Y are two jointly distributed Gaussian stochastic variables according to

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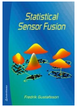

Section 7.1.3