Nonlinear Transformations Using Taylor Series Expansions

Sensor Fusion

Fredrik Gustafsson
fredrik.gustafsson@liu.se

Gustaf Hendeby
gustaf.hendeby@liu.se

Linköping University

Nonlinear Transformation (NLT) (of a stochastic variable)

In many cases it is important to perform nonlinear transformations of stochastic variables, e.g., for estimation of parameters with nonlinear measurement models.

Problem formulation: nonlinear transfomation (NLT)

Given the transform

$$z = g(u)$$

and the mean and covariance of the input,

$$\mathrm{E}(u) = \mu_u, \quad \mathrm{Cov}(u) = P_u \quad \text{(often approximated } u \sim \mathcal{N}(\mu_u, P_u))$$

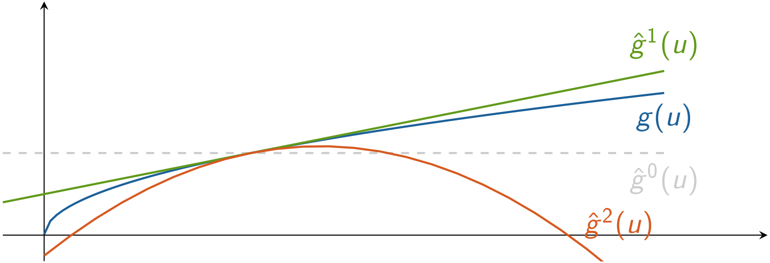
determine

$$\mathrm{E}(z) = \mu_z \quad \mathrm{Cov}(z) = P_z \quad \text{(often approximated } z \sim \mathcal{N}(\mu_z, P_z)).$$

Taylor Series Expansion

The idea behind Taylor series based transformations is:

- Approximate $g(u)$ with a low degree polynomial.
- Analytical expressions for mean and covariance can then be derived for:
  - the first order (linear) approximation, and
  - the second order approximation, if u is assumed Gaussian.



Low i degree approximations, $\hat{g}^i(u)$, of $g(u)$.

Taylor Transformations (TT) (1/2)

The first order Taylor series expansion of $z = g(u)$ around $\bar{u}$ including the rest term is

$$z = g(\bar{u}) + g'(\bar{u})(u - \bar{u}) + \underbrace{\tfrac{1}{2}(u - \bar{u})^T g''(\xi)(u - \bar{u})}_{r(\xi)},$$

where ξ is in the neighborhood of $\bar{u}$.

Taylor Transformation order 1 (TT1)

Compute the requested moments assuming $r(\xi) \equiv 0$ yields:

$$\mu_z^{TT1} = \mathrm{E}(g(\mu_u) + g'(\mu_u)(u - \mu_u)) = g(\mu_u)$$
$$P_u^{TT1} = \mathrm{Cov}(g(\mu_u) + g'(\mu_u)(u - \mu_u)) = g'(\mu_u)^T P_u g'(\mu_u)$$

Taylor Transformations (TT) (2/2)

Taylor Transformation order 2 (TT2)

Compute the requested moments assuming $\xi = \mu_x$ (hence, include also the quadratic term):

$$
\begin{aligned}
\mu_z^{TT2} &= \mathrm{E}\big(g(\mu_u) + g'(\mu_u)(u-\mu_u) + \tfrac{1}{2}(u-\mu_u)^T g''(\mu_u)(u-\mu_u)\big) \\
&= g(\mu_u) + \tfrac{1}{2}[\mathrm{tr}(g_i''(\mu_u)P_u)]_i \\
&= \mu_z^{TT1} + \tfrac{1}{2}[\mathrm{tr}(g_i''(\mu_u)P_u)]_i \\
P_z^{TT2} &= \mathrm{Cov}\big(g(\mu_u) + g'(\mu_u)(u-\mu_u) + \tfrac{1}{2}(u-\mu_u)^T g''(\mu_u)(u-\mu_u)\big) \\
&= g'(\mu_u)P_u\big(g'(\mu_u)\big)^T + \tfrac{1}{2}\Big[\mathrm{tr}(P_u g_i''(\mu_u) P_u g_j''(\mu_u))\Big]_{ij} \\
&= P_u^{TT1} + \tfrac{1}{2}\Big[\mathrm{tr}(P_u g_i''(\mu_u) P_u g_j''(\mu_u))\Big]_{ij}
\end{aligned}
$$

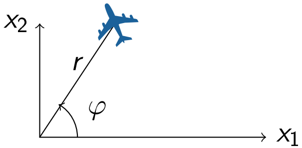
Radar Example: observations

Sensor model:

$$y = (r, \varphi)^T + e = h(x_1, x_2) + e,$$

$$r = \sqrt{x_1^2 + x_2^2} + e_r,$$

$$\varphi = \text{arctan2}(x_1, x_2) + e_\varphi.$$



Direct approach, inverting the observation model

$$x = h^{-1}(y - e),$$

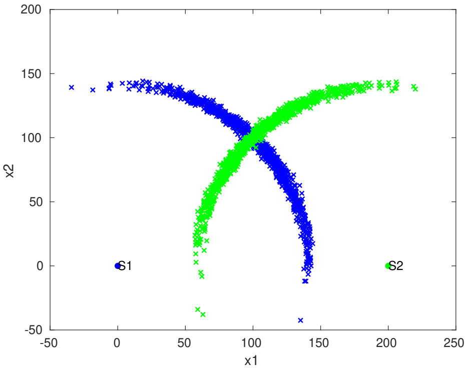
$$x_1 = y_1 \cos(y_2) = (r - e_r) \sin(\varphi - e_\varphi),$$

$$x_2 = y_1 \sin(y_2) = (r - e_r) \cos(\varphi - e_\varphi).$$

What is the covariance of $\hat{x} = h^{-1}(y)$?

Radar Example: Monte Carlo samples

- Generate measurements of range and bearing.
- Invert $x = h^{-1}(y)$ for each sample.
- Banana shaped distribution of estimates.



Matlab (SigSys):

```
hinv= @(R, Phi, p) [p(1) + R * cos(Phi);
                    p(2) + R * sin(Phi)];

R1 = ndist(100 * sqrt(2), 5);
Phi1 = ndist(pi/4, 0.1);
p1 = [0; 0];

R2 = ndist(100 * sqrt(2), 5);
Phi2 = ndist(3 * pi/4, 0.1);
p2 = [200; 0];

xhat1 = hinv(R1, Phi1, p1);
xhat2 = hinv(R2, Phi2, p2);

plot(p1(1), p1(2), '.b',...
     p2(1), p2(2), '.g',...
     'markersize', 15);
hold on;
text(p1(1), p1(2), 'S1');
text(p2(1), p2(2), 'S2');

plot2(xhat1, xhat2, 'legend', 'off');
```

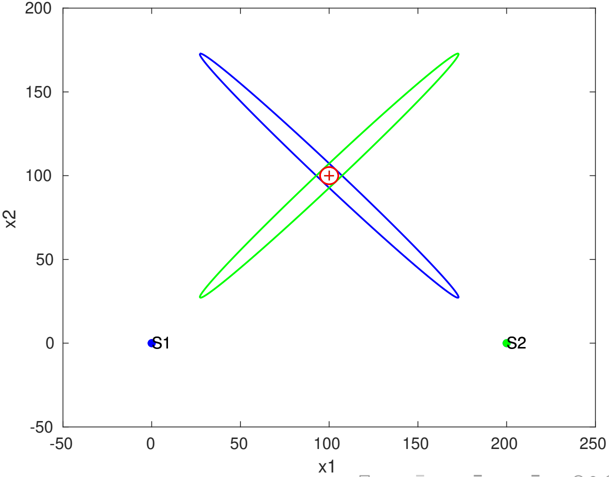
Radar Example: TT1

Gauss approximation formula (based on linearizing $h_k^{-1}(x)$) applied to the banana transformation gives too optimistic result (since higher order terms are neglected).

```
y1 = [R1; Phi1];  y2 = [R2; Phi2];
hinv = @(y, p) [p(1) + y(1,:).*cos(y(2,:));
                p(2) + y(1,:).*sin(y(2,:))];
Nhat1 = tt1eval(y1, hinv, p1)
Nhat2 = tt1eval(y2, hinv, p2)
xhat = fusion(Nhat1, Nhat2)
plot(p1(1), p1(2), '.b',...
     p2(1), p2(2), '.g', 'markersize', 15);
hold on;
text(p1(1), p1(2), 'S1');
text(p2(1), p2(2), 'S2');
plot2(Nhat1, Nhat2, xhat, 'legend', 'off');
```

Output:

```
Nhat1 =
N([100;100],[1e+03,-997;-997,1e+03])
Nhat2 =
N([100;100],[1e+03,998;998,1e+03])
xhat =
N([100;100],[4.99,-2.18e-08;-2.18e-08,4.99])
```



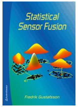
Summary: Taylor series expansion NLT

Nonlinear transformations (NLT) of stochastic variables are used to find the distribution of $z = g(u)$, given the mean and covariance of u.

Taylor series expansion based methods approximate the function using a Taylor series and then compute the mean and covariance for the simpler approximate function.

TT1: Use a first order Taylor series expansion.

TT2: Use a second order Taylor series expansion.



Section 3.4.3