The Sensor Fusion Formula

Sensor Fusion

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Repetition WLS

Linear model:

$$
\begin{aligned}
y_k &= H_k x + e_k, \quad \mathrm{Cov}(e_k) = R_k, \quad k = 1, \dots, N, \\
\mathbf{y} &= \mathbf{H}x + \mathbf{e}, \qquad \mathrm{Cov}(\mathbf{e}) = \mathbf{R}.
\end{aligned}
$$

WLS loss function

$$
V^{WLS}(x) = \sum_{k=1}^{N} (y_k - H_k x)^T R_k^{-1} (y_k - H_k x) = (\mathbf{y} - \mathbf{H}x)^T \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H}x).
$$

WLS solution

$$
\begin{aligned}
\hat{x} &= \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} \sum_{k=1}^{N} H_k^T R_k^{-1} y_k = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{R}^{-1} \mathbf{y}, \\
P &= \left(\sum_{k=1}^{N} H_k^T R_k^{-1} H_k \right)^{-1} = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1}.
\end{aligned}
$$

WLS Special Case

Consider the special case of WLS with $N = 2$ and $H_k = I$:

$$y_1 = x + e_1, \quad \mathrm{Cov}(e_1) = R_1,$$
$$y_2 = x + e_2, \quad \mathrm{Cov}(e_2) = R_2.$$

WLS solution

$$\hat{x} = \left(R_1^{-1} + R_2^{-1}\right)^{-1} \left(R_1^{-1} y_1 + R_2^{-1} y_2\right),$$
$$P = \left(R_1^{-1} + R_2^{-1}\right)^{-1}.$$

Sanity check: if $R_1 = R_2 = R$, then

$$\hat{x} = \frac{1}{2}(y_1 + y_2),$$
$$P = \frac{1}{2}R.$$

The fusion formula

Suppose now we have two estimates $\hat{x}_1$ and $\hat{x}_2$ of the parameter x, with covariances P_1 and P_2, respectively. Identify

$$\hat{x}_1 = y_1, \quad \hat{x}_2 = y_2,$$
$$P_1 = R_1, \quad P_2 = R_2.$$

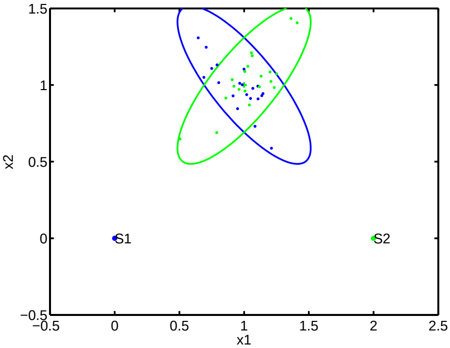
The *fusion formula* for two independent estimates can then be stated as

$$\mathrm{E}(\hat{x}_1) = \mathrm{E}(\hat{x}_2) = x, \quad \mathrm{Cov}(\hat{x}_1) = P_1, \quad \mathrm{Cov}(\hat{x}_2) = P_2 \Rightarrow$$
$$P = \left(P_1^{-1} + P_2^{-1}\right)^{-1}.$$
$$\hat{x} = P\left(P_1^{-1}\hat{x}_1 + P_2^{-1}\hat{x}_2\right).$$

Simple sensor network example, continued

Code for triangulation in *Signal and Systems Lab*:

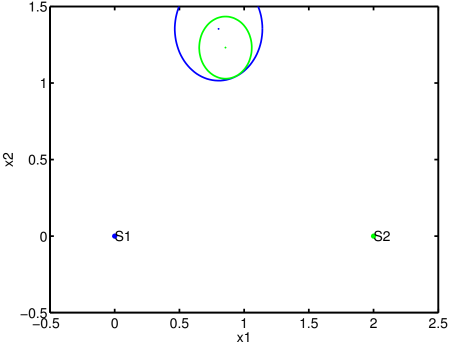
```
p1=[0;0];
p2=[2;0];
x=[1;1];
X1=ndist(x,0.1*[1 -0.8;-0.8 1]);
X2=ndist(x,0.1*[1 0.8;0.8 1]);
plot2(X1,X2)
```



Sensor network example, continued

Apply the fusion formula to merge the two estimates (the two ellipses become a smaller ellipse)

```
X3=fusion(X1,X2);    % WLS
X4=0.5*X1+0.5*X2;    % LS
plot2(X4,X3)
```

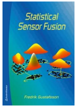


Summary Fusion

The fusion formula for two independent estimates is

$$
\begin{aligned}
\mathrm{E}(\hat{x}_1) = \mathrm{E}(\hat{x}_2) &= x, \\
\mathrm{Cov}(\hat{x}_1) = P_1, \quad \mathrm{Cov}(\hat{x}_2) &= P_2 \Rightarrow \\
\hat{x} &= P\left(P_1^{-1}\hat{x}_1 + P_2^{-1}\hat{x}_2\right), \\
P &= \left(P_1^{-1} + P_2^{-1}\right)^{-1}.
\end{aligned}
$$

If the estimates are not independent, P will be larger than indicated with this formula. then use *safe fusion*.



Sections 2.3.1–2.3.4