Continuous Time Motion Models

Sensor Fusion

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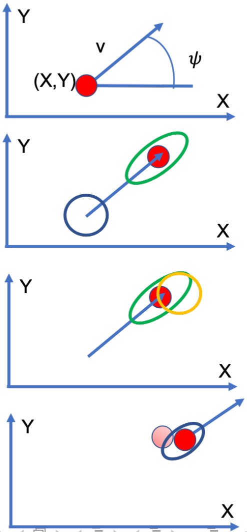
Models for Filtering

Object tracking illustration:

1. The model must be able to predict where the object turns up at the next time. For instance, the speed and heading may be part of the state vector.
2. In the time update of the filter, the current estimated position (blue circle) is translated to a new position (green ellipse) according to the estimated velocity, taking small possible manoeuvres into account (as process noise)
3. At the new position, a new sensor observation becomes available (orange circle).
4. The filter incorporates the sensor fusion formula automatically to combine the prediction with the measurement to a new position (blue ellipse).



Purpose

Provide examples of common motion models and modelling methodology

- Physics give continuous time model, filters require (linear or nonlinear) discrete time model.
- Separate lecture on how to convert a continuous time model to discrete time (discretization) that can be used in a filter.
- When modeling phenomena i nature, physical relations usually result in a time continuous dynamic model with time discrete measurements.
- This lecture, focus on continuous time state space model of the form $\dot{x} = Ax + Bu$ for linear systems and $\dot{x} = a(x, u)$ for nonlinear systems.
- Examples of basic models for navigation (sensors on the moving platform) and tracking (sensors in the infrastructure).

Example 1: Newton's II law

Linear motion governed by Newtons II law,
$F = ma = m\ddot{X}$.

To get a state space model, introduce the state vector x

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} p \\ v \end{pmatrix} \quad \Rightarrow \quad \dot{x} = \begin{pmatrix} v \\ a \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{F}{m} \end{pmatrix}$$

$$\Rightarrow$$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & \frac{1}{m} \end{pmatrix}^T u = Ax + Bu$$

Measurement:

$$y_t = \begin{pmatrix} 1 & 0 \end{pmatrix} x + e_t$$

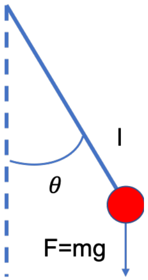
Example 2: Pendulum

The ordinary differential equation (ODE) for a pendulum is

$$\ell\ddot{\theta} + \frac{b}{2m}\dot{\theta} + g\sin\theta = 0$$

where b is the damping.
To get a state space model, let

$$x = \begin{pmatrix} \theta \\ \omega \end{pmatrix} \Rightarrow \dot{x} = \begin{pmatrix} \omega \\ -\frac{g}{\ell}\sin\theta - \frac{b}{2m\ell}\omega \end{pmatrix}.$$



Using a first order Taylor expansion, the linearized model around $\theta = 0$ is obtained

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell} & \frac{-b}{2m} \end{pmatrix} x.$$

Navigation Models

- Sensors are placed on the platform
- Usually inertial measurements (accelerometers, gyroscopes), magnetometers (compass) and speedometers.
- 2D orientation (course, or yaw rate) much easier than 3D orientation, which is covered in a separate lecture.
- Rotation in 2D is modeled by the course ψ, the simplest model being

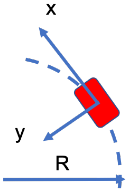
$$\dot{\psi}(t) = w(t),$$

where $w(t)$ is the input (usually unknown and modelled as noise)

Coordinated Turns in 2D Body Coordinates

A coordinated turn model aims at modelling straight as well as circular trajectories. The most basic motion equations for course rate $\dot{\psi}$, longitudinal speed v_x, lateral speed v_y and radius R of the circle (or its inverse R^{-1} which becomes zero for straight lines!)

$$\begin{aligned}
\dot{\psi} &= \frac{v_x}{R} = v_x R^{-1}, \\
a_y &= \frac{v_x^2}{R} = v_x^2 R^{-1} = v_x \dot{\psi}, \\
a_x &= \dot{v}_x - v_y \frac{v_x}{R} = \dot{v}_x - v_y v_x R^{-1} = \dot{v}_x - v_y \dot{\psi}.
\end{aligned}$$



Neglecting lateral speed (no skidding) $x = (v_x, \psi)^T$ and $u = (a_x, \dot{\psi})$ (longitudinal accelerometer and course gyroscope), we can use the state space model

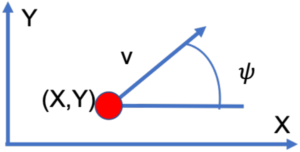
$$\dot{x} = Ax + Bu = 0 \cdot x + \begin{pmatrix} a_x \\ \dot{\psi} \end{pmatrix}$$

and then compute $R^{-1} = \dot{\psi}/v_x$ or as a_y/v_x^2.

Odometry 2D Body Coordinates

A more common model for 2D navigation is based on odometry, where the input $u = (v_x, \dot{\psi})^T$ consists of speed and course rate, and the state is $x = (X, Y, \psi)^T$

$$\begin{aligned}\dot{X} &= v^x \cos(\psi)\\ \dot{Y} &= v^x \sin(\psi)\\ \dot{\psi}_t &= 0\end{aligned}$$



This ODE has an explicit solution

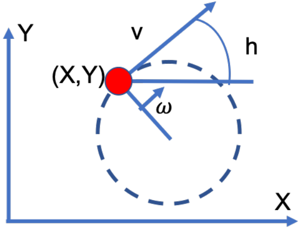
$$\begin{aligned}X_t &= X_0 + \int_0^t v_\tau^x \cos(\psi_\tau)\, d\tau\\ Y_t &= Y_0 + \int_0^t v_\tau^x \sin(\psi_\tau)\, d\tau\\ \psi_t &= \psi_0 + \int_0^t \dot{\psi}_\tau\, d\tau.\end{aligned}$$

Coordinated Turns in World Coordinates

Looking at a moving object from sensors in the infrastructure, every motion can be locally approximated well with a circular path (where a straight motion is a special case).

Coordinated turn (CT) describes a circular motion.
Heading h can be seen as the same as course ψ before.
Note that the circle is described in world coordinates here, not in a local coordinate system.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{h} \\ \dot{v} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} v\cos(h) \\ v\sin(h) \\ \omega \\ 0 \\ 0 \end{pmatrix}$$



Summary

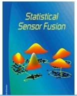
- Standard models in global coordinates:
  - Translation $p_t^{(m)} = w_t^p$.
  - 2D orientation for heading $h_t^{(m)} = w_t^h$.
  - Coordinated turn model

$$\dot{X} = v^X \qquad \dot{Y} = v^Y$$
$$\dot{v}^X = -\omega v^Y \qquad \dot{v}^Y = \omega v^X$$
$$\dot{\omega} = 0.$$

- Standard models in local coordinates (x, y, ψ):
  - Odometry and dead reckoning for (x, y, ψ)

$$X_t = X_0 + \int_0^t v_\tau^x \cos(\psi_\tau)\, d\tau \qquad Y_t = Y_0 + \int_0^t v_\tau^x \sin(\psi_\tau)\, d\tau$$
$$\psi_t = \psi_0 + \int_0^t \dot{\psi}_\tau \, d\tau.$$

  - Inertial models for $(\dot{\psi}, a_y, a_x, \dots)$.



Chapter 13–13.1, 13.4